(41)等比級數

等比級數的數列定義如下:

a_1、a_2、$a_3 \cdots a_n$

a、ar、$ar^2 \cdots ar^{n-1}$

因此，在一個等比級數內，

$a_i = a_{i-1}r$ …… (1)

$\frac{a_{i+1}}{a_i} = \mathrm{r}$ …… (2)

$a_i = \mathrm{a}r^{i-1}$ …… (3)

以下都是等比級數的例子

(1) $a = 2$，$r = 2$

2，4，8，16，32，64

(2) $a = 1$，$r = \frac{1}{2}$

$1，\frac{1}{2}，\frac{1}{4}，\frac{1}{8}，\frac{1}{16}，\frac{1}{32}，\frac{1}{64}$

(3) $a = 1$，$r = -2$

1，-2，4，-8，16，-32，64

(4) $a = 3$，$r = \frac{1}{3}$

$3，1，\frac{1}{3}，\frac{1}{9}，\frac{1}{27}，\frac{1}{81}，\frac{1}{243}$

(5) $a = 2$，$r = 3$，第 5 項是什麼?

$a_5 = \mathrm{a}r^{5-1} = 2(3)^4 = 2 \times 81 = 162$

(6) $a=1$，$r=-\frac{1}{2}$，第 5 項是什麼?

$$a_5 = \mathrm{a}r^{5-1} = 1\times(-\frac{1}{2})^4 = 1\times\frac{1}{16} = \frac{1}{16}$$

(7) $a=1$，$r=-\frac{1}{2}$，第 6 項是什麼?

$$a_6 = \mathrm{a}r^{6-1} = 1\times(-\frac{1}{2})^5 = -\frac{1}{32}$$

(8) $\sqrt{2}-1$，1，a，b是一等比級數，求r，a和b

$$r = \frac{a_{i+1}}{a_i} = \frac{a_2}{a_1} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2}+1$$

$$a = 1\times r = \sqrt{2}+1$$

$$b = ar = (\sqrt{2}+1)^2$$

以下是這個等比級數

$$\sqrt{2}-1，1，\sqrt{2}+1，(\sqrt{2}+1)^2$$

(9) $\frac{1}{\sqrt{5}-1}$，$\frac{1}{4}$，a，b是一等比級數，求r，a和b

$$r = \frac{a_{i+1}}{a_i} = \frac{a_2}{a_1} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{5}-1}} = \frac{\sqrt{5}-1}{4}$$

$$\therefore a = \frac{1}{4}\times\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{16}$$

$$b = ar = \frac{\sqrt{5}-1}{16}\times\frac{\sqrt{5}-1}{4} = \frac{(\sqrt{5}-1)^2}{64} = \frac{6-2\sqrt{5}}{64}$$

等比級數的和

令$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rS_n = ar + ar^2 + \cdots + ar^n$

$\therefore\ S_n - rS_n = a - ar^n$

$S_n(1-\mathrm{r}) = \mathrm{a}(1-r^n)$

$S_n = \dfrac{a(1-r^n)}{1-r} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$

$(10) a = 1$ ，$r = 2$ ，$n = 5$

$S_n = 1 + 2 + 4 + 8 + 16 = 31$

用公式(4)

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-2^5)}{1-2} = \frac{1(1-32)}{-1} = \frac{-31}{-1} = 31$$

$(11) a = 2$ ，$r = \frac{1}{2}$ ，$n = 6$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{2(1-(\frac{1}{2})^6)}{1-\frac{1}{2}} = \frac{2(1-\frac{1}{64})}{\frac{1}{2}} = 4(1-\frac{1}{64}) = 4(\frac{63}{64}) = \frac{63}{16}$$

同學可以自行驗證答案的正確性