# (19)更多的三角函數公式

## 半角公式

1. $sin\frac{\theta}{2} = \sqrt{\frac{1-cos\,\theta}{2}}$

   證明：

   $\because cos\,2\,\theta = 1 - 2\,sin^2\,\theta$

   $\therefore cos\,\theta = 1 - 2\,sin^2\frac{\theta}{2}$

   $2\,sin^2\frac{\theta}{2} = 1 - cos\,\theta$

   $sin^2\frac{\theta}{2} = \frac{1-cos\,\theta}{2}$

   $sin\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\,\theta}{2}}$

   (等號右端取正或取負，由$\frac{\theta}{2}$所在的象限決定)

2. $cos\frac{\theta}{2} = \pm\sqrt{\frac{1+cos\,\theta}{2}}$

   (等號右端取正或取負，由$\frac{\theta}{2}$所在的象限決定)

   證明的方法同上

3. $tan\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\,\theta}{1+cos\,\theta}}$

   利用以上兩個公式即可證明

4. $\theta = 90°$，求$sin\frac{\theta}{2}$和$cos\frac{\theta}{2}$。

$$cos\,\theta = cos\,90° = 0$$

$$sin\frac{\theta}{2} = sin\,45° = \sqrt{\frac{1-cos\,90°}{2}} = \sqrt{\frac{1-0}{2}} = \frac{1}{\sqrt{2}}$$

$$cos\frac{\theta}{2} = cos\,45° = \sqrt{\frac{1+cos\,90°}{2}} = \sqrt{\frac{1+0}{2}} = \frac{1}{\sqrt{2}}$$

5. $\theta = 60°$，求$sin\frac{\theta}{2}$和$cos\frac{\theta}{2}$。

$$cos\,\theta = cos\,60° = \frac{1}{2}$$

$$sin\frac{\theta}{2} = sin\,30° = \sqrt{\frac{1-cos\,60°}{2}} = \sqrt{\frac{1-\frac{1}{2}}{2}} = \frac{1}{2}$$

$$cos\frac{\theta}{2} = cos\,30° = \sqrt{\frac{1+cos\,60°}{2}} = \sqrt{\frac{1+\frac{1}{2}}{2}} = \frac{\sqrt{3}}{2}$$

**和積互化公式**

6. $2sin\alpha cos\beta = sin(\alpha+\beta) + sin(\alpha-\beta)$

證明：

$sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta \ldots(1)$
$sin(\alpha-\beta) = sin\alpha cos\beta - cos\alpha sin\beta \ldots(2)$

(1) + (2)得

$2sin\alpha cos\beta = sin(\alpha+\beta) + sin(\alpha-\beta)$

7. $2cos\alpha sin\beta = sin(\alpha+\beta) - sin(\alpha-\beta)$

證明方法同 6.

8. $2cos\alpha cos\beta = cos(\alpha+\beta) + cos(\alpha-\beta)$

證明：

$cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta \ldots(1)$
$cos(\alpha-\beta) = cos\alpha cos\beta + sin\alpha sin\beta \ldots(2)$

(1) +(2)得

$2cos\alpha cos\beta = cos(\alpha+\beta) + cos(\alpha-\beta)$

9. $2sin\alpha sin\beta = cos(\alpha-\beta) - cos(\alpha+\beta)$

證明方法同 8.

10. 已知 $\alpha=30^\circ$， $\beta=60^\circ$，驗證 $2sin\alpha cos\beta = sin(\alpha+\beta) + sin(\alpha-\beta)$

過程：

$sin\alpha = sin30^\circ = \frac{1}{2}$

$cos\beta = cos60^\circ = \frac{1}{2}$

$2sin\alpha cos\beta = \frac{1}{2}$

$sin(\alpha+\beta) = sin(30^\circ + 60^\circ) = sin90^\circ = 1$

$sin(\alpha-\beta) = sin(30^\circ - 60^\circ) = sin(-30^\circ) = -\frac{1}{2}$

$sin(\alpha+\beta) + sin(\alpha-\beta) = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$

因此 $\alpha=30^\circ$， $\beta=60^\circ$ 時，$2sin\alpha cos\beta = sin(\alpha+\beta) + sin(\alpha-\beta)$

11.已知 $\alpha=60^\circ$， $\beta=30^\circ$，驗證$2cos\alpha cos\beta = cos(\alpha+\beta)+cos(\alpha-\beta)$

過程：

$cos\alpha = cos60^\circ = \frac{1}{2}$

$cos\beta = cos30^\circ = \frac{\sqrt{3}}{2}$

$2cos\alpha cos\beta = \frac{\sqrt{3}}{2}$

$cos(\alpha+\beta) = cos(60^\circ+30^\circ) = cos90^\circ = 0$

$cos(\alpha-\beta) = cos(60^\circ-30^\circ) = cos30^\circ = \frac{\sqrt{3}}{2}$

$cos(\alpha+\beta)+cos(\alpha-\beta) = 0+\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

因此$\alpha = 60^\circ$，$\beta = 30^\circ$時，$2cos\alpha cos\beta = cos(\alpha+\beta)+cos(\alpha-\beta)$

**互餘函數的疊合**

12.設 $a,b$ 為實數，且$a^2+b^2 \neq 0$，則：

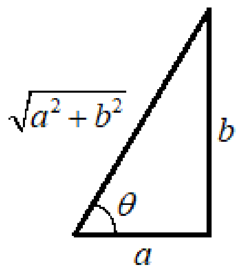
$y = a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\theta)$

$\theta$滿足：$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$、$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$

證明：

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\frac{a}{\sqrt{a^2+b^2}}\sin x + \sqrt{a^2+b^2}\frac{b}{\sqrt{a^2+b^2}}\cos x$$

我們可以想像有一個直角三角形



在三角形中：

$sin\,\theta = \frac{b}{\sqrt{a^2+b^2}}$、$cos\,\theta = \frac{a}{\sqrt{a^2+b^2}}$

所以

$$
\begin{aligned}
& a\,sin\,x + b\,cos\,x \\
& = \sqrt{a^2+b^2}\frac{a}{\sqrt{a^2+b^2}}sin\,x + \sqrt{a^2+b^2}\frac{b}{\sqrt{a^2+b^2}}cos\,x \\
& = \sqrt{a^2+b^2}\,cos\,\theta\,sin\,x + \sqrt{a^2+b^2}\,sin\,\theta\,cos\,x \\
& = \sqrt{a^2+b^2}(cos\,\theta\,sin\,x + sin\,\theta\,cos\,x) \\
& = \sqrt{a^2+b^2}(sin\,x + \theta)
\end{aligned}
$$

可寫成函數形式：$y = a\,sin\,x + b\,cos\,x = \sqrt{a^2+b^2}\,sin(\,x+\theta)$

13.將$y=\frac{\sqrt{3}}{2}sin\,x+\frac{1}{2}cos\,x$寫成$y=sin(\,x+\theta)$的形式，並以$x=60$°驗證。

過程：

$$a=\frac{\sqrt{3}}{2}$$
$$b=\frac{1}{2}$$
$$\sqrt{a^2+b^2}=\sqrt{(\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$$
$$sin\,\theta=\frac{b}{\sqrt{a^2+b^2}}=\frac{1}{2}$$
$$cos\,\theta=\frac{a}{\sqrt{a^2+b^2}}=\frac{\sqrt{3}}{2}$$

滿足此條件的$\theta$為 30°

因此$y=\frac{\sqrt{3}}{2}sin\,x+\frac{1}{2}cos\,x$可寫成$y=sin(\,x+30°)$

$x=60$°時

$$\frac{\sqrt{3}}{2}sin\,60°+\frac{1}{2}cos\,60°=\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\frac{1}{2}\times\frac{1}{2}=1$$
$$sin(\,x+30°)=sin(\,60°+30°)=sin\,90°=1$$

可知$y=\frac{\sqrt{3}}{2}sin\,x+\frac{1}{2}cos\,x$=$sin(\,x+30°)$在$x=60$°時正確。

14. 將$y = \sin x + \cos x$寫成$y = \sin(x + \theta)$的形式，並以$x = 45^\circ$驗證。

過程：

$a = 1$
$b = 1$

$\sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}} = \dfrac{1}{\sqrt{2}}$

$\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}} = \dfrac{1}{\sqrt{2}}$

滿足此條件的$\theta$為 $45^\circ$

因此$y = \sin x + \cos x$可寫成$y = \sqrt{2}\sin(x + 45^\circ)$

$x = 45^\circ$時

$\sin 45^\circ + \cos 45^\circ = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

$\sin(x + 45^\circ) = \sqrt{2}\sin(45^\circ + 45^\circ) = \sqrt{2}\sin 90^\circ = \sqrt{2}$

可知$y = \sin x + \cos x = \sqrt{2}\sin(x + 45^\circ)$在$x = 45^\circ$時正確。

和差化積公式

15. $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$,求證:

我們已知 $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

令 $A = \alpha + \beta$、$B = \alpha - \beta$

$\alpha = \frac{A+B}{2}$、$\beta = \frac{A-B}{2}$

$\therefore \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

故得證

16.用同樣的方法,可以得到

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

17.驗證 $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

令$A = 0^o$、$B = 60^o$

$$\sin A + \sin B = \sin 0^o + \sin 60^o = \frac{\sqrt{3}}{2}$$
$$\sin\frac{A+B}{2} = \sin 30^o = \frac{1}{2}$$
$$\cos\frac{A-B}{2} = \sin(-30^o) = \frac{\sqrt{3}}{2}$$
$\because 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\therefore \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

18.驗證 $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

令 A=$60^o$, B=$0^o$

$$\cos A = \cos 60^o = \frac{1}{2}$$
$$\cos B = \cos 0^o = 1$$
$\therefore \cos A + \cos B = \frac{1}{2} + 1 = \frac{3}{2}$

$$\cos\frac{A+B}{2} = \cos 30^o = \frac{\sqrt{3}}{2}$$
$$\cos\frac{A-B}{2} = \cos 30^o = \frac{\sqrt{3}}{2}$$
$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 2\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} = \frac{3}{2}$$
$\therefore \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$