(17) 半角的三角函數

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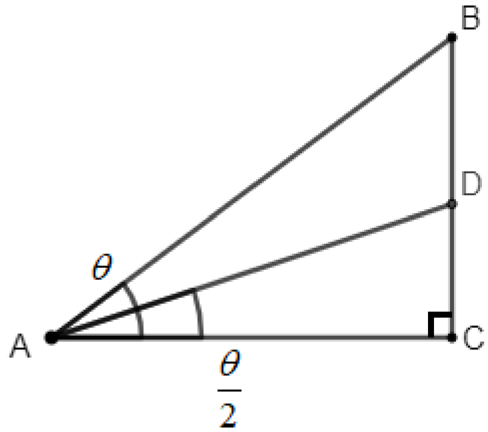


$\triangle ABC$ 中，$\angle C$ 為直角，D 點在$\overline{BC}$上，且$\overline{AD}$為$\angle A$ 的角平分線。

$\angle A = \theta$，$\angle CAD = \frac{\theta}{2}$

證明$sin\frac{\theta}{2} = \frac{sin\,\theta}{\sqrt{2}\times\sqrt{1+cos\,\theta}}$

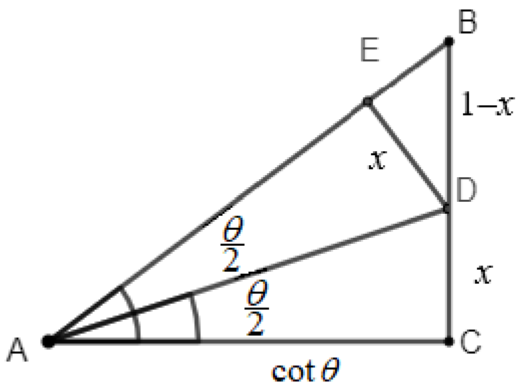
過程：

假設$\overline{BC}=1$，

$\frac{\overline{AC}}{\overline{BC}} = cot\,\theta$，$\overline{AC} = cot\,\theta$

在$\overline{AB}$上取一點 E，且$\overline{DE}\perp\overline{AB}$。

設$\overline{DC}=x$，則$\overline{BD}=1-x$



$\because \angle DAC = \angle DAE = \frac{\theta}{2}$，$\overline{AD} = \overline{AD}$，$\angle ADC = \angle ADE = 90^\circ - \frac{\theta}{2}$

$\therefore \triangle AED \cong \triangle ACD$ (ASA)

$\therefore \overline{ED} = \overline{DC} = x$

$\angle B = 90° - \angle A = 90° - \theta$

由$\triangle BED$ 可以看出$\sin B = \frac{x}{1-x}$

$\sin B = \sin(90° - \theta) = \cos\theta$

$\therefore \cos\theta = \frac{x}{1-x}$

$x = \frac{\cos\theta}{1+\cos\theta}$…(1)

$\triangle ADC$ 中

$$\overline{AD}^2 = x^2 + \cot^2\theta$$

$$\overline{AD}^2 = \left(\frac{\cos\theta}{1+\cos\theta}\right)^2 + \cot^2\theta$$

$$\overline{AD}^2 = \frac{\cos^2\theta}{(1+\cos\theta)^2} + \frac{\cos^2\theta}{\sin^2\theta}$$

$$\overline{AD}^2 = \frac{\cos^2\theta\sin^2\theta + \cos^2\theta(1+\cos\theta)^2}{(1+\cos\theta)^2\sin^2\theta}$$

$$\overline{AD}^2 = \frac{\cos^2\theta(\sin^2\theta + (1+\cos\theta)^2)}{(1+\cos\theta)^2\sin^2\theta}$$

$$\overline{AD}^2 = \frac{\cos^2\theta(\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta)}{(1+\cos\theta)^2\sin^2\theta}$$

$$\overline{AD}^2 = \frac{\cos^2\theta(2 + 2\cos\theta)}{(1+\cos\theta)^2\sin^2\theta}$$

$$\overline{AD}^2 = \frac{2\cos^2\theta}{(1+\cos\theta)\sin^2\theta}$$

$\therefore \overline{AD} = \frac{\sqrt{2}\cos\theta}{\sin\theta\sqrt{1+\cos\theta}}$…(2)

$$\sin\frac{\theta}{2} = \frac{x}{\overline{AD}}$$

$$\sin\frac{\theta}{2} = \frac{\frac{\cos\theta}{1+\cos\theta}}{\frac{\sqrt{2}\cos\theta}{\sin\theta\sqrt{1+\cos\theta}}}$$

$$\sin\frac{\theta}{2} = \frac{\frac{\cos\theta}{1+\cos\theta}}{\frac{\sqrt{2}\cos\theta}{\sin\theta\sqrt{1+\cos\theta}}}$$

$$\sin\frac{\theta}{2} = \frac{\cos\theta}{1+\cos\theta} \times \frac{\sin\theta\sqrt{1+\cos\theta}}{\sqrt{2}\cos\theta}$$

$$\sin\frac{\theta}{2} = \frac{\sin\theta}{\sqrt{2}\sqrt{1+\cos\theta}}$$

例 1

θ =60 °，求$\sin\frac{\theta}{2}$

詳解：

$$\sin\frac{\theta}{2} = \sin\frac{60°}{2} = \frac{\sin 60°}{\sqrt{2}\sqrt{1+\cos 60°}} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{2}\sqrt{1+\frac{1}{2}}} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} = \frac{1}{2}$$

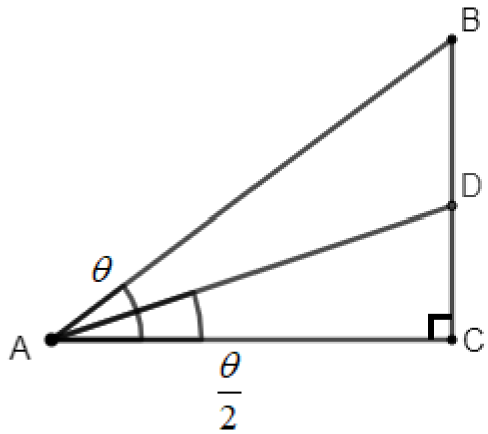
例 2

θ =90 °，求$\sin\frac{\theta}{2}$

詳解：

$$\sin\frac{\theta}{2} = \sin\frac{90°}{2} = \frac{\sin 90°}{\sqrt{2}\sqrt{1+\cos 90°}} = \frac{1}{\sqrt{2}\sqrt{1+0}} = \frac{1}{\sqrt{2}}$$

我們找到了求$sin\frac{\theta}{2}$的方式，就可以很容易地找到求$cos\frac{\theta}{2}$的方式。



求$sin\frac{\theta}{2}$時我們已經知道$\overline{AD}=\frac{\sqrt{2}\cos\theta}{\sin\theta\sqrt{1+\cos\theta}}$

$$cos\frac{\theta}{2}=\frac{\overline{AC}}{\overline{AD}}$$

$$cos\frac{\theta}{2}=\frac{\cot\theta}{\frac{\sqrt{2}\cos\theta}{\sin\theta\sqrt{1+\cos\theta}}}$$

$$cos\frac{\theta}{2}=\frac{\cot\theta}{\frac{\sqrt{2}\cot\theta}{\sqrt{1+\cos\theta}}}$$

$$cos\frac{\theta}{2}=\frac{\sqrt{1+\cos\theta}}{\sqrt{2}}$$

我們已經知道$sin\frac{\theta}{2}=\frac{\sin\theta}{\sqrt{2}\sqrt{1+\cos\theta}}$、$cos\frac{\theta}{2}=\frac{\sqrt{1+\cos\theta}}{\sqrt{2}}$

所以$sin\frac{\theta}{2}\times cos\frac{\theta}{2}=\frac{\sin\theta}{2}$

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

我們也可以寫成$sin2\theta=2sin\theta cos\theta$

例 3

$\theta = 30^{\circ}$，求$sin2\theta$

詳解：

$$sin\,2\theta = sin(\,2\times 30^{\circ}) = 2\,sin\,30^{\circ}cos\,30^{\circ} = 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

例 4

$\theta = 45^{\circ}$，求$sin2\theta$

詳解：

$$sin2\theta = sin(\,2\times 45^{\circ}) = 2\,sin\,45^{\circ}cos\,45^{\circ} = 2\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = 1$$

例 5

$\theta = 15^{\circ}$，求$sin2\theta$

詳解：

之前我們學過

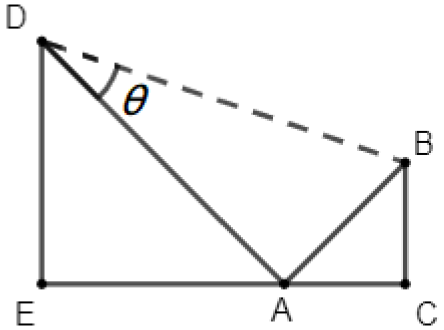
$$sin\,15^{\circ} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$cos\,15^{\circ} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$sin2\theta = sin(\,2\times 15^{\circ}) = 2\,sin\,15^{\circ}cos\,15^{\circ} = 2\times\frac{\sqrt{6}-\sqrt{2}}{4}\times\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{6-2}{8} = \frac{1}{2}$$

三角形中角的三角函數

1.



上圖中，$\angle E = \angle C = 90^\circ$，$\overline{EA} = \overline{ED} = 2$，$\overline{CA} = \overline{CB} = 1$，求$sin\theta$之值。

詳解：

$\overline{CA} = \overline{CB}$，$\angle C=90^\circ$

$\therefore \angle BAC=45^\circ$

同理$\angle DAE=45^\circ$

$\therefore \angle DAB = 180^\circ - 45^\circ - 45^\circ = 90^\circ$

$\therefore sin\,\theta = \frac{\overline{AB}}{\overline{BD}}$

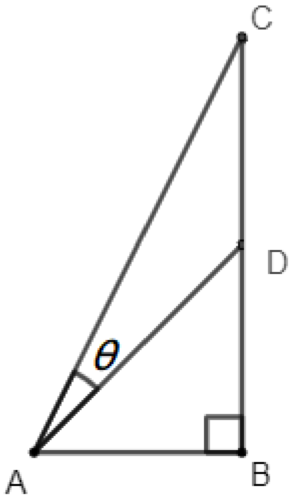
$\overline{AB} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\overline{AD} = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\overline{BD} = \sqrt{\sqrt{2}^2 + \sqrt{8}^2} = \sqrt{10}$

$$sin\,\theta = \frac{\overline{AB}}{\overline{BD}} = \frac{\sqrt{2}}{\sqrt{10}} = \frac{1}{\sqrt{5}}$$

2.

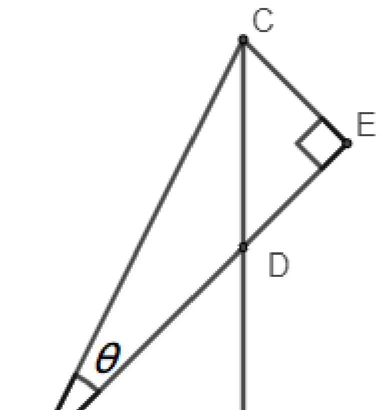


上圖中，$\overline{AB}$ =1，$\overline{BC}$ =2，$\overline{CD}=\overline{DB}$ =1，求$sin\theta$之值。

詳解：

$\overline{AC}=\sqrt{\overline{AB}^2+\overline{BC}^2}=\sqrt{1^2+2^2}=\sqrt{5}$

延長$\overline{AD}$至 E 點，且$\overline{AE}\perp\overline{EC}$



$\overline{AB}=\overline{DB}$ =1，$\triangle ABD$ 是等腰直角三角形

$\angle BAD=\angle BDA=45^\circ$

$\angle CDE=\angle BDA=45^\circ$(對頂角相等)

$\angle DCE=90^\circ-\ \ 5^\circ=45^\circ$

$\angle DCE=\angle CDE$

$\therefore \overline{CE} = \overline{DE}$

$$\overline{CD}^2 = 1 = \overline{CE}^2 + \overline{DE}^2 = 2\overline{CE}^2$$

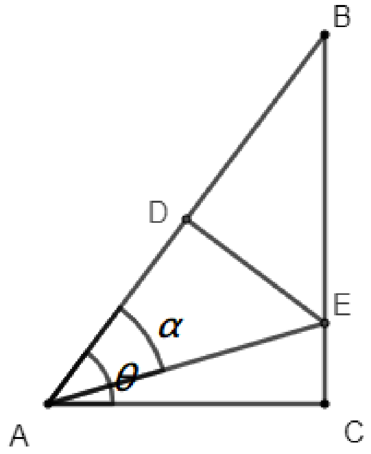
$$\overline{CE}^2 = \frac{1}{2}$$

$$\overline{CE} = \frac{1}{\sqrt{2}}$$

$\triangle ACE$ 中

$$\sin\theta = \frac{\overline{CE}}{\overline{AC}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{5}} = \frac{1}{\sqrt{10}}$$

3.



上圖中，$\angle BAC=\theta$，$\angle BAE=\alpha$，$\overline{DE}$是$\overline{AB}$的中垂線，將$sin\alpha$用θ表示。

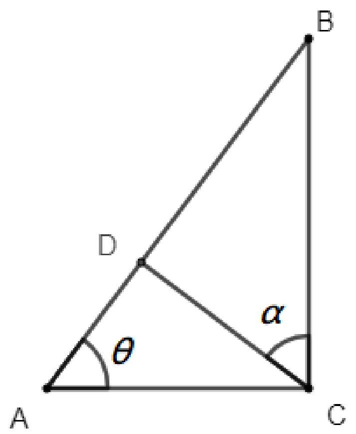
詳解：

依題意$\triangle ADE \cong \triangle BDE\ (SAS)$

$\therefore \alpha = \angle EBA = \angle CBA = 90^\circ - \theta$

$sin\alpha = sin(90^\circ - \theta) = cos\theta$

4.



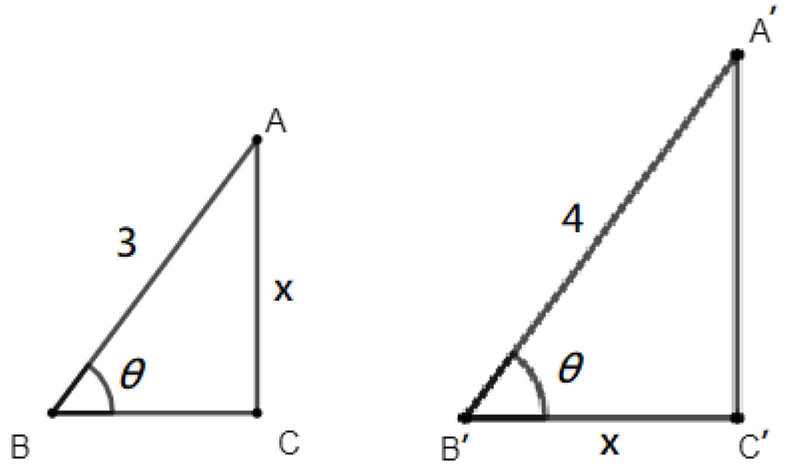
上圖中，$\angle BCA = 90^\circ$，$\overline{AB} \perp \overline{DC}$，$\angle BAC = \theta$，$\angle BCD = \alpha$，將$sin\alpha$用$\theta$表示。

詳解：

$\triangle BCD$ 中，$\angle B + \angle\alpha = 90^\circ$；$\triangle ABC$ 中，$\angle B + \angle\theta = 90^\circ$

可知$\alpha = \theta$，$sin\alpha = sin\theta$

5.



上圖兩中角形中，$\angle C = \angle C' = 90^\circ$，求 x 之值。

詳解：

$\triangle ABC$ 中，$\sin\theta = \frac{x}{3}$…(1)

$\triangle A'B'C'$中

$\overline{A'C'} = \sqrt{4^2 - x^2} = \sqrt{16 - x^2}$

$\sin\theta = \frac{\sqrt{16-x^2}}{4}$…(2)

由(1)(2)可知

$$\frac{x}{3} = \frac{\sqrt{16-x^2}}{4}$$

$$4x = 3\sqrt{16-x^2}$$

$$16x^2 = 9(16 - x^2)$$

$$16x^2 = 144 - 9x^2$$

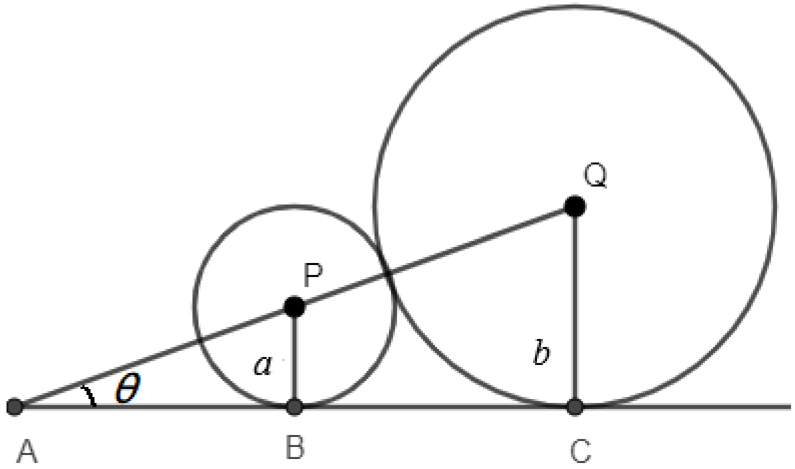
$$25x^2 = 144$$

$$x^2 = \frac{144}{25}$$

$$x = \frac{12}{5}$$

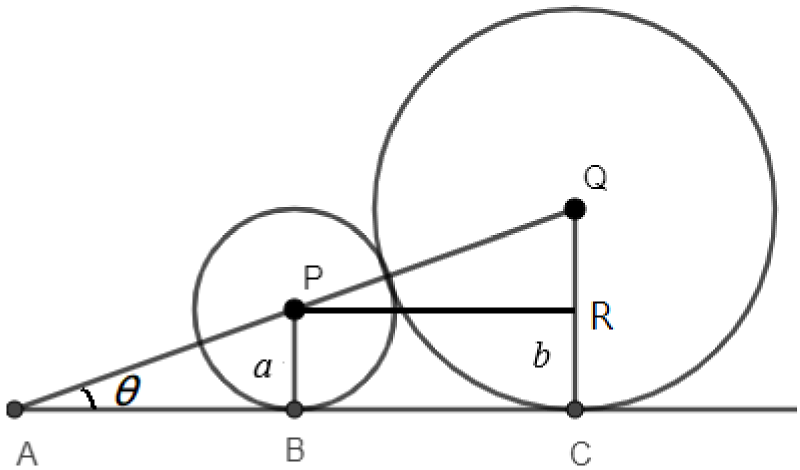
(長度為正數，取正值)

6.



如上圖，P、Q 兩圓相切，半徑分別為a、b，$a < b$，$\overline{AC}$為 P、Q 的切線，切點分別為 B、C。求$tan\theta$。

詳解：



從 P 做$\overline{BC}$之平行線，與$\overline{QC}$交於 R 點。

$\angle QPR = \theta$ (同位角相等)

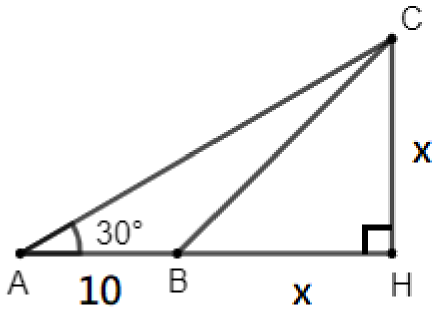
$\overline{PQ} = \mathrm{a} + \mathrm{b}$

$\overline{QR} = \mathrm{b} - \mathrm{a}$

$\overline{PR} = \sqrt{(a+b)^2 - (b-a)^2} = \sqrt{4ab} = 2\sqrt{ab}$

$$tan\,\theta = \frac{\overline{QR}}{\overline{PR}} = \frac{b-a}{2\sqrt{ab}}$$

7.



如上圖，求 x 之值。

詳解：

$\triangle AHC$ 為 30°-60°-90°的直角三角形

$$\sin A = \sin 30° = \frac{1}{2} = \frac{x}{\overline{AC}}$$

故 $\overline{AC} = 2x$

$$cos\,30° = \frac{\sqrt{3}}{2} = \frac{\overline{AH}}{\overline{AC}} = \frac{10 + x}{2x}$$

$$\frac{\sqrt{3}}{2} = \frac{10 + x}{2x}$$

$$\sqrt{3} = \frac{10 + x}{x}$$

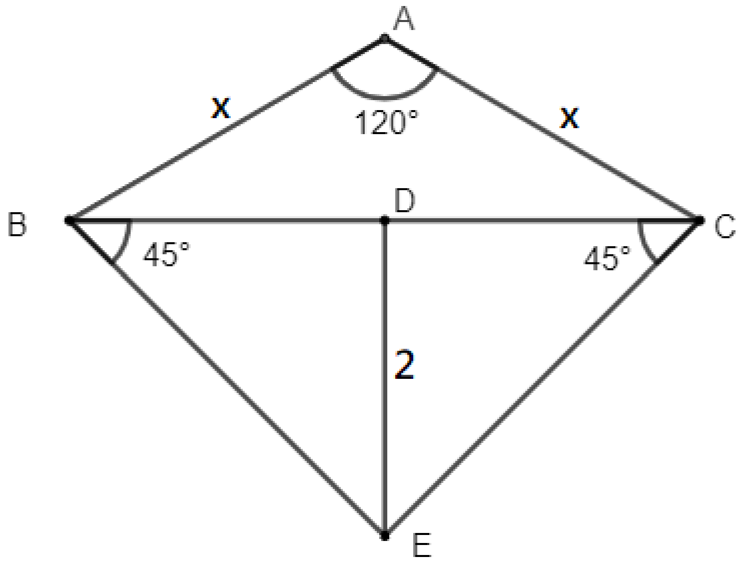
$$\sqrt{3}x = 10 + x$$

$$\sqrt{3}x - x = 10$$

$$(\sqrt{3} - 1)x = 10$$

$$x = \frac{10}{\sqrt{3 - 1}}$$

8.



如上圖，求 x 之值。

詳解：

$$\overline{BC}=\sqrt{x^2+x^2-2x^2\cos 120^\circ}=\sqrt{2x^2-2x^2\left(-\frac{1}{2}\right)}=\sqrt{3x^2}=\sqrt{3}x$$

$$\overline{BD}=\frac{1}{2}\overline{BC}=\frac{\sqrt{3}}{2}x$$

因為 $\angle DBE=45^\circ$，$\overline{BD}=\overline{DE}=\frac{\sqrt{3}}{2}x=2$

$x=\dfrac{4}{\sqrt{3}}$