(15) 三角總複習 1

1. 在ΔABC中，求證：$\sin A = \sin(B + C)$

【證明】$\because \angle A = 180° - (\angle B + \angle C)$

$\therefore \sin A = \sin(180° - (B + C)) = \sin(B + C)$

2. 在ΔABC中，求證：$\sin\frac{B+C-A}{2} = \cos A$

【證明】

$\because B + C = 180° - A$

$\therefore \sin\dfrac{B + C - A}{2}$

$= \sin\dfrac{180° - 2A}{2}$

$= \sin(90° - A)$

$= \cos A$

3. 已知$\tan\theta = a$，求$\cos\theta$

【解答】$\because 1 + \tan^2\theta = \sec^2\theta$

$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$

$\therefore \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+a^2}$

$\cos\theta = \frac{1}{\sqrt{1+a^2}}$

4. 求證：$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}=2\,sec^2\theta$

【證明】$\because \frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$

$$=\frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}=\frac{2}{1-sin^2\theta}=\frac{2}{cos^2\theta}=2\,sec^2\theta$$

5. 求證：$(1-\tan^4\theta)\,cos^2\theta+\tan^2\theta=1$

【證明】 $(1-\tan^4\theta)\,cos^2\theta+\tan^2\theta$

$$=(1-\tan^2\theta)(1+\tan^2\theta)\,cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)\,sec^2\theta\,cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)\frac{1}{cos^2\theta}cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)+\tan^2\theta$$
$$=1$$

6. 求證：$\frac{cos\,\theta}{1+\sin\theta}+\frac{cos\,\theta}{1-\sin\theta}=2\,sec\,\theta$

【證明】 $\frac{cos\,\theta}{1+\sin\theta}+\frac{cos\,\theta}{1-\sin\theta}$

$$=\frac{cos\,\theta\,(1-\sin\theta)+cos\,\theta\,(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$
$$=\frac{2\,cos\,\theta}{1-sin^2\theta}$$
$$=\frac{2\,cos\,\theta}{cos^2\theta}$$
$$=\frac{2}{cos\,\theta}$$
$$=2\,sec\,\theta$$

7. 求證：

$$\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}=2\sec\theta$$

【證明】

$$\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}$$
$$=\frac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}$$
$$=\frac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$$
$$=\frac{1+1+2\sin\theta}{(1+\sin\theta)\cos\theta}$$
$$=\frac{2(1+\sin\theta)}{(1+\sin\theta)\cos\theta}$$
$$=\frac{2}{\cos\theta}=2\sec\theta$$

8. 求證：

$$\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}+\frac{\tan\theta+1}{\tan\theta-1}=\frac{2}{\sin^2\theta-\cos^2\theta}$$

【證明】

$$\because\frac{\tan\theta+1}{\tan\theta-1}=\frac{\frac{\sin\theta}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}-1}=\frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\cos\theta}}=\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$$
$$\therefore\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}+\frac{\tan\theta+1}{\tan\theta-1}$$
$$=\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$$
$$=\frac{(\sin\theta-\cos\theta)^2+(\sin\theta+\cos\theta)^2}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$$
$$=\frac{2(\sin^2\theta+\cos^2\theta)}{\sin^2\theta-\cos^2\theta}$$
$$=\frac{2}{\sin^2\theta-\cos^2\theta}$$

9. 求證：$(\sin\theta-\frac{1}{\sin\theta})^2-(tan\,\theta-\frac{1}{tan\,\theta})^2+(cos\,\theta-\frac{1}{cos\,\theta})^2=1$

【證明】$(\sin\theta-\frac{1}{\sin\theta})^2-(tan\,\theta-\frac{1}{tan\,\theta})^2+(cos\,\theta-\frac{1}{cos\,\theta})^2$

$=(\sin^2\theta+\frac{1}{\sin^2\theta}-2)-(tan^2\,\theta+\frac{1}{tan^2\,\theta}-2)+(cos^2\,\theta+\frac{1}{cos^2\,\theta}-2)$

$$=\sin^2\theta+\frac{1}{\sin^2\theta}-2-\frac{\sin^2\theta}{cos^2\,\theta}-\frac{cos^2\,\theta}{\sin^2\theta}+2+cos^2\,\theta+\frac{1}{cos^2\,\theta}-2$$
$$=(\sin^2\theta+cos^2\,\theta)+\frac{1-cos^2\,\theta}{\sin^2\theta}+\frac{1-\sin^2\theta}{cos^2\,\theta}-2$$
$$=1+\frac{sin^2\,\theta}{\sin^2\theta}+\frac{cos^2\,\theta}{cos^2\,\theta}-2$$
$$=1+1+1-2$$
$$=1$$

10. 求證：

$$\frac{1+sin\,\theta}{1-\sin\theta}=(\frac{cos\,\theta}{1-\sin\theta})^2$$

【證明】

$$\frac{1+sin\,\theta}{1-\sin\theta}$$
$$=\frac{(1+sin\,\theta)(1-\sin\theta)}{(1-\sin\theta)(1-\sin\theta)}$$
$$=\frac{1-\sin^2\theta}{(1-\sin\theta)^2}$$
$$=\frac{cos^2\,\theta}{(1-\sin\theta)^2}$$
$$=(\frac{cos\,\theta}{1-\sin\theta})^2$$

11. 求證：$(1+\sin\theta+\cos\theta)(1+\sin\theta-\cos\theta)=2\sin\theta\,(1+\sin\theta)$

【證明】$(1+\sin\theta+\cos\theta)(1+\sin\theta-\cos\theta)$

$$=(1+\sin\theta)^2-\cos^2\theta$$
$$=(1+\sin\theta)^2-(1-\sin^2\theta)$$
$$=(1+\sin\theta)^2-(1+\sin\theta)(1-\sin\theta)$$
$$=(1+\sin\theta)(1+\sin\theta-1+\sin\theta)$$
$$=2\sin\theta\,(1+\sin\theta)$$

12. 求證：

$$\frac{(1+tan\theta)^2}{(1-tan\theta)^2}=\frac{1+2sin\theta cos\theta}{1-2sin\theta cos\theta}$$

【證明】

$$\frac{(1+tan\theta)^2}{(1-tan\theta)^2}$$
$$=\frac{\left(1+\frac{sin\theta}{cos\theta}\right)^2}{\left(1-\frac{sin\theta}{cos\theta}\right)^2}$$
$$=\frac{\left(\frac{cos\theta+sin\theta}{cos\theta}\right)^2}{\left(\frac{cos\theta-sin\theta}{cos\theta}\right)^2}$$
$$=\left(\frac{cos\theta+sin\theta}{cos\theta-sin\theta}\right)^2$$
$$=\frac{cos^2\theta+sin^2\theta+2sin\theta cos\theta}{cos^2\theta+sin^2\theta-2sin\theta cos\theta}$$
$$=\frac{1+2sin\theta cos\theta}{1-2sin\theta cos\theta}$$

13. 已知$\begin{cases} x = \sin\theta - 2 \ldots\ldots (1) \\ y = \cos\theta + 2 \ldots\ldots (2) \end{cases}$，求證：$(x+2)^2 + (y-2)^2 = 1$

【證明】由(1)得 $x + 2 = \sin\theta$
由(2)得 $y - 2 = \cos\theta$
$\therefore (x+2)^2 + (y-2)^2 = \sin^2\theta + \cos^2\theta = 1$

14. 求證：$1 - \sin\theta + \sin^2\theta - \cos^2\theta = \sin\theta\,(2\sin\theta - 1)$

【證明】$1 - \sin\theta + \sin^2\theta - \cos^2\theta$
$= \sin^2\theta + \cos^2\theta - \sin\theta + \sin^2\theta - \cos^2\theta$
$= 2\sin^2\theta - \sin\theta$
$= \sin\theta\,(2\sin\theta - 1)$

15. 求證：$1 + (\sin\theta - \cos\theta)^2 = 2(1 - \sin\theta\cos\theta)$

【證明】$1 + (\sin\theta - \cos\theta)^2$
$= 1 + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta$
$= 1 + 1 - 2\sin\theta\cos\theta$
$= 2 - 2\sin\theta\cos\theta$
$= 2(1 - \sin\theta\cos\theta)$

16. 已知$\begin{cases} \sin\theta + \cos\theta = -k \ldots\ldots (1) \\ \sin\theta\cos\theta = k \ldots\ldots\ldots\ldots (2) \end{cases}$，求 k

【解答】由(1)得

$$(1)^2 \Rightarrow (\sin\theta + \cos\theta)^2 = (-k)^2$$
$$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = k^2$$
$$\Rightarrow 1 + 2\sin\theta\cos\theta = k^2 \ldots\ldots. (3)$$

將(2)代入(3)得

$$1 + 2k = k^2$$
$$k^2 - 2k - 1 = 0$$
$$k = \frac{2 \pm \sqrt{4+4}}{2}$$
$$= \frac{2 \pm 2\sqrt{2}}{2}$$
$$= 1 \pm \sqrt{2}$$

$\mathrm{k} = 1 + \sqrt{2} > 1$，不合(2)

$$\because -1 < \sin\theta < 1$$
$$-1 < \cos\theta < 1$$
$$\Rightarrow -1 < \sin\theta\cos\theta < 1$$
$$\Rightarrow -1 < k < 1$$
$$\therefore k = 1 - \sqrt{2}$$

17. 求證：$(\cos A\,(1 + \tan A))^2 + (\cos A\,(1 - \tan A))^2 = 2$

【證明】$(\cos A\,(1 + \tan A))^2 + (\cos A\,(1 - \tan A))^2$

$$= \cos^2 A\,(1 + \tan A)^2 + \cos^2 A\,(1 - \tan A)^2$$
$$= \cos^2 A\,(1 + 2\tan A + \tan^2 A) + \cos^2 A\,(1 - 2\tan A + \tan^2 A)$$
$$= \cos^2 A\,(1 + 2\tan A + \tan^2 A + 1 - 2\tan A + \tan^2 A)$$
$$= \cos^2 A\,(2 + 2\tan^2 A)$$
$$= 2\cos^2 A\,(1 + \tan^2 A)$$
$$= 2\cos^2 A \sec^2 A$$
$$= 2$$