# (75)數列和的運算

1. 試證$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

用歸納法

$n = 1$, $\sum_{k=1}^{n} k^2 = 1$

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = \frac{2 \times 3}{6} = 1$$

$\therefore n = 1, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$成立

假設$k = n$,此公式成立，我們要證明當$k = n + 1$時，此公式仍然成立

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \ldots (1)$$

$$\therefore \sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(2n+3)(n+2)}{6}$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \ldots (2)$$

比較(1)和(2)，可以看出當$k = n + 1$時，此公式仍然成立的

例$n = 3$

$$\sum_{k=1}^{3} k^2 = 1 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{3(4)(7)}{6} = \frac{12 \times 7}{6} = 14$$

2. $\sum_{k=1}^{n} k^3 = (\frac{n(n+1)}{2})^2$

$$k = 1, \sum_{k=1}^{1} k^3 = 1$$

$$\frac{n(n+1)}{2} = \frac{1(2)}{2} = 1$$

∴當$k = 1$時，公式成立

假設$k = n$時，此公式成立，我們要證明當$k = n + 1$時，此公式仍然成立

也就是說，我們要證明$\sum_{k=1}^{n+1} k^3 = (\frac{(n+1)(n+2)}{2})^2$

$$\sum_{k=1}^{n} k^3 = (\frac{n(n+1)}{2})^2 = \frac{n^2(n^2+2n+1)}{4} = \frac{n^4+2n^3+n^2}{4}$$

$$\sum_{k=1}^{n+1} k^3 = \frac{n^4+2n^3+n^2}{4} + (n+1)^3$$

$$= \frac{n^4+2n^3+n^2+4n^3+12n^2+12n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

但是，$\left((n+1)(n+2)\right)^2 = (n^2+3n+2)^2 = n^4+6n^3+13n^2+12n+4$

$$\therefore \sum_{k=1}^{n+1} k^3 = (\frac{(n+1)(n+2)}{2})^2$$

∴公式成立

例$n = 3$

$$\sum_{k=1}^{n} k^3 = 1 + 2^2 + 3^2 = 1 + 8 + 27 = 36 = 6^2$$

$$n = 3, (\frac{(n+1)(n+2)}{2})^2 = (\frac{3(4)}{2})^2 = 6^2$$

3. 若$a_1 = 1, a_{i+1} = 2S_n + 1$, 求證$a_i = 3^{i-1}$

令$n = 1$

$a_{n+1} = a_2$

$S_n = S_1 = 1$

$\therefore 2S_n + 1 = 2 \times 1 + 1 = 3$

$\therefore a_2 = 3 = 3^{2-1}$

$\therefore n = 1$時，$a_n = 3^{n-1}$成立

現在去求$S_n$，$a_1, a_2, a_3, \dots, a_n$為一等比級數，$r = 3, a_1 = 1$

$$\therefore S_n = \frac{3^n - 1}{3 - 1}$$

$$2S_n + 1 = \frac{2(3^n - 1)}{2} + 1 = 3^n - 1 + 1 = 3^n$$

$\therefore a_{n+1} = 3^n$

4. 求$\sum_{k=1}^{n} k(k+1)$

$$\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{2n(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(n+2)}{3}$$

例$n = 4$

$$\sum_{k=1}^{n} k(k+1) = 2 + (2^2 + 2) + (3^3 + 3) + (4^4 + 4)$$

$$= 2 + 6 + 12 + 20$$

$$= 40$$

$$\frac{n(n+1)(n+2)}{3} = \frac{4(5)(6)}{3} = 40$$

5. 求$sin^2(1°) + sin^2(2°) + sin^2(3°) + \cdots + sin^2(89°)$

$S_n = sin^2(1°) + sin^2(2°) + sin^2(3°) + \cdots + sin^2(89°)$

$S_n = sin^2(89°) + sin^2(88°) + sin^2(87°) + \cdots + sin^2(1°)$

$\therefore 2S_n = (sin^2(1°) + sin^2(89°)) + \left(sin^2(2°) + sin^2(88°)\right) + \cdots$

$+(sin^2(89°) + sin^2(1°))$

$\because \sin(90° - \theta) = cos\theta$

$\therefore 2S_n = (sin^2(1°) + cos^2(1°)) + (sin^2(2°) + cos^2(2°)) + \cdots$

$+(sin^2(89°) + cos^2(89°))$

=89

6. $f(x)=\frac{4^x}{4^x+2}$

求$S_n=f\left(\frac{1}{a}\right)+f\left(\frac{2}{a}\right)+\cdots+f(\frac{a-1}{a})$

$$S_n=f\left(\frac{1}{a}\right)+f\left(\frac{2}{a}\right)+\cdots+f\left(\frac{a-1}{a}\right)\dots(1)$$

$$S_n=f\left(\frac{a-1}{a}\right)+f\left(\frac{a-2}{a}\right)+\cdots+f\left(\frac{1}{a}\right)\dots(2)$$

$$2S_n=\left(f\left(\frac{1}{a}\right)+f\left(\frac{a-1}{a}\right)\right)+\left(f\left(\frac{2}{a}\right)+f\left(\frac{a-2}{a}\right)\right)+\cdots+(f\left(\frac{a-1}{a}\right)+f\left(\frac{1}{a}\right))$$

$$f\left(\frac{i}{a}\right)+f\left(\frac{a-i}{a}\right)=\frac{4^{\frac{i}{a}}}{4^{\frac{i}{a}}+2}+\frac{4^{\frac{a-i}{a}}}{4^{\frac{a-i}{a}}+2}$$

$$=\frac{4^{\frac{i}{a}}(4^{a-i}+2)+4^{\frac{a-i}{a}}\left(4^{\frac{i}{a}}+2\right))}{(4^{\frac{i}{a}}+2)(4^{\frac{a-i}{a}}+2)}$$

$$=\frac{4^{\frac{a}{a}}+2\cdot4^{\frac{i}{a}}+4^1+2\cdot4^{\frac{a-i}{a}}}{4^1+2\cdot4^{\frac{i}{a}}+4^1+2\cdot4^{\frac{a-i}{a}}}$$

$$=1$$

$$\therefore 2S_n=a\times1=a$$

$$S_n=\frac{a}{2}$$

7. 求$S_n=\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n\times(n+1)}$

$$S_n=\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n\times(n+1)}$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$$=\frac{n+1-1}{n+1}$$

$$=\frac{n}{n+1}$$