(74)等差和等比的進階題目

1. $< a_n >$是一等差級數，則$< \alpha a_n >$也是等差級數

   例1,3,5,7,9,11,13是一等差級數

   3,9,15,21,27,33,39也是等差級數

   證明:

   $< a_n >$是一等差級數，故$a_i = a_i + (i-1)d$

   $\alpha a_i = \alpha a_i + \alpha(i-1)d = \alpha a_i + (i-1)\alpha d$

   令$\alpha a_i = a_i',\ \alpha a_1 = a_1',\ \alpha d = d'$

   則$a_i' = a_1' + d'$

   故$< \alpha a_n >$是一等差級數

   同理可證$< a_n + b >$也是一等差級數

2. $< a_n >$是一等比級數，則$< a_n^k >$也是等比級數

   例1,2,4,8,16,32為一等比級數，$r = 2$

   $k = 2$　1,4,16,64,256,1024為一等比級數，$\mathrm{r} = 2^2 = 4$

   因為$< a_n >$為一等比級數，故$a_i = a_i r^{i-1}$

   $a_i^k = a_1^k r^{(i-1)k}$

   $a_{i+1}^k = (a_1 r^i)^k = a_1^k r^{ik}$

   $\therefore \frac{a_{i+1}^k}{a_i^k} = \frac{a_1^k r^{ik}}{a_1^k r^{(i-1)k}} = r^{ik-ik+k} = r^k$

   $\therefore < a_n^k >$為等比級數

3. 等差級數$a_1 = 1,\ d = 2$

此等差級數1,3,5,7,9,11,13,15

$S_1 = 1$

$S_2 = 1 + 3 = 4 = 2^2$

$S_3 = 4 + 5 = 9 = 3^2$

$S_4 = 9 + 7 = 16 = 4^2$

$S_5 = 16 + 9 = 25 = 5^2$

$S_6 = 25 + 11 = 36 = 6^2$

$S_7 = 36 + 13 = 49 = 7^2$

$S_8 = 49 + 15 = 64 = 8^2$

$S_1 = 1^2$

$S_2 = 4 = 2^2$

$\vdots$

假設已知

$S_i = i^2$

$$\begin{aligned} S_{i+1} &= S_i + a_{i+1} \\ &= i^2 + (a_1 + id) \\ &= i^2 + (1 + 2i) \\ &= i^2 + 2i + 1 \\ &= (i + 1)^2 \end{aligned}$$

此證

4. 承上題，$b_1 = 1,\ b_2 = S_2 - S_1, b_3 = S_3 - S_2, \cdots$, $b_i = S_i - S_{i-1}$
求證$< b_n >$是一等差級數，$b_1 = 1,\ d = 2$

由(3) $b_i = S_i - S_{i-1}$
$= i^2 - (i-1)^2$
$= i^2 - i^2 + 2i - 1$
$= 2i - 1$

$b_{i+1} = 2(i+1) - 1 = 2i + 1$

$\therefore \mathrm{d} = b_{i+1} - b_i = (2i+1) - (2i-1) = 2$

$b_1 = 1$
$b_2 = 2^2 - 1 = 4 - 1 = 3$
$b_3 = 3^2 - 2^2 = 9 - 4 = 5$
$b_4 = 4^2 - 3^2 = 16 - 9 = 7$
$b_5 = 5^2 - 4^2 = 25 - 16 = 9$
$b_6 = 6^2 - 5^2 = 36 - 25 = 11$
$b_7 = 7^2 - 6^2 = 49 - 36 = 13$
$b_8 = 8^2 - 7^2 = 64 - 49 = 15$

$< b_n >$是一等差級數

5. $< a_n >$是一等差級數，$s_i = a_1 + a_2 + \cdots + a_i, b_1 = s_1, b_2 = s_2 - s_1, \ldots,$
$s_i = s_i - s_{i-1}$,求證$b_1, b_2, \ldots, b_n$是一等差級數

$$s_{i-1} = \left(\frac{i-1}{2}\right)(2a_1 + (i-2)d) = (i-1)a_1 + \frac{(i-1)(i-2)d}{2}$$

$$s_i = \left(\frac{i}{2}\right)(2a_1 + (i-1)d) = ia_1 + \frac{i(i-1)d}{2}$$

$$\begin{aligned} \therefore b_i = s_i - s_{i-1} &= a_1 + \frac{i(i-1) - (i-1)(i-2)d}{2} \\ &= a_1 + \frac{(i-1)(i-i+2)d}{2} \\ &= a_1 + \frac{(i-1)2d}{2} \\ &= a_1 + (i-1)d \end{aligned}$$

從以上的式子可以看出$< b_n >$是一等差級數

例 等差級數 1, 4, 7, 10, 13, 16, 19

$s_1 = 1$
$s_2 = 1 + 4 = 5$
$s_3 = 5 + 7 = 12$
$s_4 = 12 + 10 = 22$
$s_5 = 22 + 13 = 35$
$s_6 = 35 + 16 = 51$
$s_7 = 51 + 19 = 70$

$b_1 = s_1 = 1$
$b_2 = s_2 - s_1 = 5 - 1 = 4$
$b_3 = s_3 - s_2 = 12 - 5 = 7$
$b_4 = s_4 - s_3 = 22 - 12 = 10$
$b_5 = s_5 - s_4 = 35 - 22 = 13$
$b_6 = s_6 - s_5 = 51 - 35 = 16$
$b_7 = s_7 - s_6 = 70 - 51 = 19$

可以看出$< b_n >$是一等差級數，而且$b_i = a_1 + (i-1)d = 1 + 3(i-1)$

6. $< a_n >$為一等比級數，求證$s_i - s_j = r^j s_{i-j}$

證明$s_i = a_1 + a_1 r + \cdots + a_1 r^{i-1}$

$s_j = a_1 + a_1 r + \cdots + a_1 r^{j-1}$

$\therefore\ s_i - s_j = a_1 r^j + a_1 r^{j+1} + \cdots + a_1 r^{i-1}$

$= r^j\left(a_1 + a_1 r + \cdots + a_1 r^{i-j-1}\right)$

$= r^j s_{i-j}$

例　等比級數$a_1 = 1, \mathrm{r} = 2$，此等比級數為1,2,4,8,16,32

$s_1 = 1$

$s_2 = 4$

$s_3 = 7$

$s_4 = 15$

$s_5 = 31$

$s_6 = 63$

$s_6 - s_3 = 63 - 7 = 56 = r^3 s_{6-3} = 8s_3 = 8 \times 7 = 56$

答案正確

7. 一等比級數，$s_3 = 7, s_6 = 63$，求r

$s_i - s_j = r^j s_{i-j}$

$s_6 - s_3 = r^3 s_{6-3} = r^3 s_3 = 7r^3$

$\because s_6 - s_3 = 63 - 7 = 56$

$\therefore 7r^3 = 56$

$r^3 = 8$

$r = 2$

8. 求$1.1+2.01+3.001+4.0001$

$1.1+2.01+3.001+4.0001$

$=(1+0.1)+(2+0.01)+(3+0.001)+(4+0.0001)$

$=(1+2+3+4)+(0.1+0.01+0.001+0.0001)$

1,2,3,4是一等差級數

$1+2+3+4=\frac{4}{2}(1+4)=\frac{4}{2}\times 5=10$

0.1,0.01,0.001,0.0001為一等比級數

$0.1+0.01+0.001+0.0001=\frac{0.1(1-(0.1)^4)}{1-0.1}=0.1111$

$\therefore 1.1+2.01+3.001+4.0001=10+0.1111=10.1111$

9. 有一等比級數，第 3 項等於 4，前二項的和是-1，求此等比級數

$a_3=ar^{3-1}=ar^2=4\ldots(1)$

$a+a\mathrm{r}=-1\ldots(2)$

$\frac{(1)}{(2)}=\frac{ar^2}{a+a\mathrm{r}}=\frac{ar^2}{a(1+r)}=\frac{r^2}{1+r}=\frac{4}{-1}=-4$

$\therefore\ \frac{r^2}{1+r}=-4$

$r^2+4r+4=0$

$(r+2)^2=0$

$r=-2\ldots(3)$

代(3)入(1) $a(-2)^2=4, a=1$

此等比級數為$1,-2,4,-8$